



TITLE:

# Clifford theory for association schemes (Research on finite groups and their representations, vertex operator algebras, and algebraic combinatorics)

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# Clifford theory for association schemes

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## 1 Introduction

Association schemes are regarded as generalizations of finite groups. So it is natural to consider the generalization to association schemes of the theory of representation of finite groups.

Let  $K$  be an algebraically closed field. Let  $G$  be a finite group,  $N$  a normal subgroup of  $G$ . The usual Clifford theory for finite groups shows that

- (CF1) the restriction of an irreducible  $KG$ -module to  $KN$  is a direct sum of  $G$ -conjugates of an irreducible  $KN$ -module  $L$  with the same multiplicities;
- (CF2) there exists a natural bijection between the set of irreducible  $KG$ -modules over  $L$  and the set of  $KT$ -modules over  $L$ , where  $T$  is the stabilizer of  $L$  in  $G$ ;
- (CF3) and there exists a natural bijection between the set of irreducible  $KT$ -modules over  $L$  and the set of irreducible modules of a generalized group algebra of  $T/N$ .

We will generalize them to association schemes. But we only consider module over the complex number field  $\mathbb{C}$ .

## 2 Adjacency algebras of association schemes

We fix some notations for association schemes.

Let  $(X, S)$  be an association scheme. We denote by  $\sigma_s$  the adjacency matrix of  $s \in S$ . The intersection number is denoted by  $p_{st}^u$  for  $s, t, u \in S$ , namely  $\sigma_s \sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$ . The valency is denoted by  $n_s$  for  $s \in S$ . An elements in the quotient scheme  $S//T$  is denoted by  $s^T$ .

### 2.1 Generalized adjacency algebras

In this section we define generalized adjacency algebras based on a definition of generalized group algebra. Details for factor sets and generalized group algebra (ring) are available in the literature [6, Chapter 2, Section 7].

Let  $G$  be a group and let  $K$  be a field. We say that  $\alpha : G \times G \rightarrow K^\times$  is a *factor set* if it satisfies the following condition:

$$\alpha(xy, z)\alpha(x, y) = \alpha(y, z)\alpha(x, yz) \text{ for all } x, y, z \in G.$$

Note that in general we can consider the action of  $G$  on  $K$ , but to simplify our arguments, we suppose that the action is trivial. Two factor sets  $\alpha$  and  $\beta$  are *cohomologous* if there exists a map  $\gamma : G \rightarrow K^\times$  such that

$$\alpha(x, y) = \beta(x, y)\gamma(x)\gamma(y)\gamma(xy)^{-1}$$

and we write  $\alpha \sim \beta$  in this case. The relation  $\sim$  is an equivalence relation on the set of factor sets. A factor set  $\alpha$  is said to be *normalized* if  $\alpha(x, 1) = \alpha(1, x) = 1$  for all  $x \in G$ . For a normalized factor set  $\alpha$ ,  $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$  also holds. For an arbitrary factor set  $\alpha$ , there exists a normalized factor set  $\beta$  such that  $\beta \sim \alpha$ .

Let  $(X, S)$  be an association scheme and let  $T$  be a strongly normal closed subset of  $S$ . Then the quotient  $S//T$  can be regarded as a finite group. Let  $\alpha : S//T \times S//T \rightarrow K^\times$  be a factor set. We define a  $K$ -algebra  $K^{(\alpha)}S = \bigoplus_{u \in S} K\sigma_u^{(\alpha)}$  with formal basis  $\{\sigma_u^{(\alpha)} \mid u \in S\}$  and multiplication

$$\sigma_u^{(\alpha)}\sigma_v^{(\alpha)} = \sum_{w \in S} p_{uv}^w \alpha(u^T, v^T) \sigma_w^{(\alpha)}.$$

The algebra  $K^{(\alpha)}S$  is called the *generalized adjacency algebra* of  $(X, S)$  over  $K$  with factor set  $\alpha$ . If the strongly normal closed subset  $T$  is trivial, then the scheme is thin and the generalized adjacency algebra is just a generalized group algebra.

## 2.2 Graded modules and simple modules

Let  $K$  be a field. Let  $(X, S)$  be a scheme and  $T$  a strongly normal closed subset of  $S$ . Then  $S//T$  is thin and we can regard it as a finite group. Then

$$KS = \bigoplus_{s^T \in S//T} K(TsT)$$

is an  $S//T$ -graded  $K$ -algebra, where  $K(TsT) = \bigoplus_{u \in TsT} K\sigma_u$ . Obviously  $(KS)_{1^T} = KT$ . We can apply Dade's theory for  $KS$ , but we restrict our attention to the case  $K = \mathbb{C}$ .

**Theorem 2.1.** [4, Theorem 3.6] For any simple  $\mathbb{C}T$ -module  $L$  and  $s \in S$ ,  $L \otimes \mathbb{C}(TsT)$  is a simple  $\mathbb{C}T$ -module or 0.

For any simple  $\mathbb{C}T$ -module  $L$ , the set of  $S//T$ -conjugates is  $\{L \otimes \mathbb{C}(TsT) \mid s \in S, L \otimes \mathbb{C}(TsT) \neq 0\}$ . We remark that there exist examples such that  $L$  and  $L'$  are  $S//T$ -conjugate simple  $\mathbb{C}T$ -modules but their dimensions are different.

### 3 Clifford Theory

First we define some notations. Let  $A$  a finite-dimensional  $K$ -algebra and let  $B$  be a subalgebra of  $A$ . For a right  $B$ -module  $L$ , the induction  $L \otimes_B A$  of  $L$  to  $A$  is denoted by  $L \uparrow^A$ . For a right  $A$ -module  $M$ , we write  $M \downarrow_B$  if  $M$  is considered as a  $B$ -module. We denote by  $\text{IRR}(A)$  the complete set of representatives of the isomorphism classes of simple  $A$ -modules. Suppose that both  $A$  and  $B$  are semisimple. For a simple  $B$ -module  $L$ , we define  $\text{IRR}(A \mid L) = \{M \in \text{IRR}(A) \mid \text{Hom}_A(L \uparrow^A, M) \neq 0\}$ .

Let  $(X, S)$  be an association scheme and let  $T$  be a closed subset of  $S$ . For a right  $\mathbb{C}T$ -module  $L$  and a right  $\mathbb{C}S$ -module  $M$ , we write  $L \uparrow^S$  and  $M \downarrow_T$  instead of  $L \uparrow^{\mathbb{C}S}$  and  $M \downarrow_{\mathbb{C}T}$ , respectively.

In the rest of this section, we fix a scheme  $(X, S)$  and its strongly normal closed subset  $T$ .

Let  $M \in \text{IRR}(\mathbb{C}S)$ . Then  $M \in \text{IRR}(\mathbb{C}S \mid L)$  for some  $L \in \text{IRR}(\mathbb{C}T)$ . Since  $M$  is a direct summand of  $L \uparrow^S$ , any simple submodule of  $M \downarrow_T$  is an  $S//T$ -conjugate of  $L$ . If  $L$  and  $L'$  are  $S//T$ -conjugate, then  $L \uparrow^S \cong L' \uparrow^S$  as  $\mathbb{C}S$ -modules. So

$$\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L, M \downarrow_T) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L', M \downarrow_T).$$

This shows the following theorem.

**Theorem 3.1.** [4, Theorem 4.1] Let  $M \in \text{IRR}(\mathbb{C}S)$ . There exists  $L \in \text{IRR}(\mathbb{C}T)$  such that  $M \in \text{IRR}(\mathbb{C}S \mid L)$ . Then there exists a positive integer  $e$  such that

$$M \downarrow_T \cong e \left( \bigoplus_{L' \in C} L' \right),$$

where  $C = \{L \otimes \mathbb{C}(TsT) \mid s \in S, L \otimes \mathbb{C}(TsT) \neq 0\}$ .

Fix a simple  $\mathbb{C}T$ -module  $L$ . Put  $U//T$  the stabilizer of  $L$  in  $S//T$ . Then

$$\bigoplus_{s^T \in S//T} L \otimes \mathbb{C}(TsT) = L \otimes_{\mathbb{C}T} \mathbb{C}S \supset L \otimes_{\mathbb{C}T} \mathbb{C}U = \bigoplus_{u^T \in U//T} L \otimes \mathbb{C}(TuT)$$

and, by Theorem 2.1,

$$\bigoplus_{u^T \in U//T} L \otimes \mathbb{C}(TuT) \cong n_{U//T} L$$

as a  $\mathbb{C}T$ -module. So  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}(L \uparrow^U, L \uparrow^U) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L, L \uparrow^U \downarrow_T) = n_{U//T}$ . On the other hand, by the Frobenius reciprocity, we have

$$\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(L \uparrow^S, L \uparrow^S) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L, L \uparrow^S \downarrow_T) = n_{U//T}.$$

So  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(L \uparrow^S, L \uparrow^S) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}(L \uparrow^U, L \uparrow^U)$ . Let  $L \uparrow^U \cong \bigoplus_i m_i M_i$  be the irreducible decomposition of  $L \uparrow^U$ , with the property that  $M_i \cong M_j$  if and only if  $i = j$ . Then

$$\begin{aligned} \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}(L \uparrow^U, L \uparrow^U) &= \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}\left(\bigoplus_i m_i M_i, \bigoplus_i m_i M_i\right) \\ &\leq \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}\left(\bigoplus_i m_i M_i \uparrow^S, \bigoplus_i m_i M_i \uparrow^S\right) \\ &= \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(L \uparrow^S, L \uparrow^S) \end{aligned}$$

This means that  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(M_i \uparrow^S, M_i \uparrow^S) = 1$  and  $M_i \uparrow^S$  is a simple  $\mathbb{C}S$ -module for every  $i$ . Also  $M_i \uparrow^S \cong M_j \uparrow^S$  if and only if  $i = j$ . Obviously  $M_i \in \text{IRR}(\mathbb{C}U|L)$  and  $M_i \uparrow^S \in \text{IRR}(\mathbb{C}S|L)$ .

Conversely, let  $N \in \text{IRR}(\mathbb{C}S|L)$ . Then  $N$  is a direct summand of  $L \uparrow^S$ . So there exists some  $M_i$  such that  $N$  is a direct summand of  $M_i \uparrow^S$ . Since  $M_i \uparrow^S$  is simple, such  $M_i$  is uniquely determined. This shows the following theorem.

**Theorem 3.2.** [4, Theorem 4.2] Fix a simple  $\mathbb{C}T$ -module  $L$ . Put  $U//T$  the stabilizer of  $L$  in  $S//T$ . Then there exists a bijection  $\tau : \text{IRR}(\mathbb{C}U|L) \rightarrow \text{IRR}(\mathbb{C}S|L)$  such that  $\tau(M) = M \uparrow^S$  and  $\tau^{-1}(N)$  is the unique direct summand of  $N \downarrow_U$  contained in  $\text{IRR}(\mathbb{C}U|L)$ .

We consider  $\text{End}_{\mathbb{C}U}(L \uparrow^U)$ . For  $u^T \in U//T$ , we define  $\rho_{u^T} \in \text{End}_{\mathbb{C}U}(L \uparrow^U)$  by  $(\rho_{u^T}(\ell))_{v^T} = \ell_{u^T v^T}$ . Then  $\text{End}_{\mathbb{C}U}(L \uparrow^U) = \bigoplus_{u^T \in U//T} \mathbb{C} \rho_{u^T}$  and this is a  $U//T$ -graded algebra ([3, Section 4]). The multiplication is  $\rho_{u^T} \rho_{v^T} = \alpha(u^T, v^T) \rho_{u^T v^T}$  and this defines a factor set  $\alpha$ . Now  $\text{End}_{\mathbb{C}U}(L \uparrow^U) \cong \mathbb{C}^{(\alpha)}(U//T)$  is a generalized group algebra with factor set  $\alpha$ .

**Proposition 3.3.** [5, Theorem 3.1] Under the above assumptions, the irreducible  $\mathbb{C}T$ -module  $L$  is extensible to a  $\mathbb{C}^{(\alpha^{-1})}U$ -module ( $\mathbb{C}^{(\alpha^{-1})}U$  is the generalized adjacency algebra with factor set  $\alpha^{-1}$ ). The action is given by  $\ell \sigma_u^{(\alpha^{-1})} = \rho_{(u^T)^{-1}}(\ell \sigma_u)$  for  $\ell \in L$  and  $u \in U$ .

We denote by  $\tilde{L}$  the extension of  $L$  to  $\mathbb{C}^{(\alpha^{-1})}U$ . Since  $L$  is a simple  $\mathbb{C}T$ -module,  $\tilde{L}$  is a simple  $\mathbb{C}^{(\alpha^{-1})}U$ -module.

If  $M$  is an irreducible  $\mathbb{C}^{(\alpha)}(U//T)$ -module, then  $\tilde{L} \otimes_{\mathbb{C}} M$  is an irreducible  $\mathbb{C}U$ -module and is in  $\text{IRR}(\mathbb{C}U|L)$ . So we can define a map  $\mu : \text{IRR}(\mathbb{C}^{(\alpha)}(U//T)) \rightarrow \text{IRR}(\mathbb{C}U|L)$  by

$$\mu(M) = \tilde{L} \otimes_{\mathbb{C}} M.$$

Then  $\mu$  is a bijection. This shows the following theorem.

**Theorem 3.4.** [5, Theorem 3.6] Let  $(X, S)$  be an association scheme, let  $T$  be a strongly normal closed subset, and let  $L$  be an irreducible  $\mathbb{C}T$ -module. Let  $U//T$  be the stabilizer of  $L$  in  $S//T$ . Then  $L$  is extensible to a  $\mathbb{C}^{(\alpha^{-1})}U$ -module  $\tilde{L}$  and the map  $\mu : \text{IRR}(\mathbb{C}^{(\alpha)}(U//T)) \rightarrow \text{IRR}(\mathbb{C}U|L)$  defined by  $\mu(M) = \tilde{L} \otimes_{\mathbb{C}} M$  is a bijection.

## References

- [1] E.C. Dade, Clifford theory for group-graded rings, *J. Reine Angew. Math.* 369 (1986) 40–86.
- [2] A. Hanaki, Character products of association schemes, *J. Algebra* 283 (2005) 596–603.
- [3] A. Hanaki, Clifford theory for association schemes, *J. Algebra* 321 (2009) 1686–1695.
- [4] A. Hanaki, Representations of finite association schemes, *Eur. J. Combin.* 30 (2009) 1477–1496.
- [5] A. Hanaki, Y. Miyazaki, Clifford theory for association schemes II, *J. Algebra*, vol.408, pp.109–114, 2014.
- [6] H. Nagao, Y. Tsushima, *Representations of Finite Groups*, Academic Press, Boston, MA, 1989.
- [7] P.-H. Zieschang, *An Algebraic Approach to Association Schemes*, Lecture Notes in Mathematics, Vol. 1628, Springer-Verlag, Berlin, 1996.